Some Lucas and Fibonacci number determinants

Johann Cigler

We compute the following determinants which are related to the Lucas numbers L_n and the Fibonacci numbers F_n.

$$\det\left(\binom{2i+1}{i+j+1}+\binom{2i+3}{i+j+2}\right)_{i,j=0}^{n-1}=L_{2n+1} \tag{1}$$

$$\det\left(\binom{2i+2}{i+j+2}+\binom{2i+4}{i+j+3}\right)_{i,j=0}^{n-1}=\sum_{j=0}^{n}L_{2j+1}=L_{2n+2}-2, \tag{2}$$

$$\det\left(\binom{2i}{i+j}-\binom{2i}{i+j+1}+\binom{2i+2}{i+j+1}-\binom{2i+2}{i+j+2}\right)_{i,j=0}^{n-1}=F_{2n+1}, \tag{3}$$

and

$$\det\left(\binom{2i+1}{i+j+1}-\binom{2i+1}{i+j+2}+\binom{2i+3}{i+j+2}-\binom{2i+3}{i+j+3}\right)_{i.j=0}^{n-1}=F_{2n+2}. \tag{4}$$

Identity (2) has been conjectured by Tony Foster in a Facebook group about Pascal's triangle.

Let us first state some well-known general results.

Lemma 1

Let $\left(a(i,j)\right)_{i,j=0}^{n-1}$ *be an invertible matrix with determinant* 1 *and* $\left(b(i,j)\right)_{i,j=0}^{n-1}$ *its inverse.*

The numbers $b(n,0)$ *satisfy*

$$\sum_{j=0}^{n}a(n,j)b(j,0)=[n=0] \tag{5}$$

for each n *and are uniquely determined by these equations.*

Cramer's rule gives

$$\det\left(a(i+1,j)\right)_{i,j=0}^{n-1}=(-1)^{n}b(n,0). \tag{6}$$

The symbol $[P]$ gives 1 if property P is true and 0 else.

Let L_n denote the Lucas numbers and let $l_n = L_n$ for $n > 0$ and $l_0 = 1$.

Then l_n satisfies $l_n = l_{n-1} + t_{n-2} l_{n-2}$ with $l_0 = 1$ and $l_1 = 1$, where $t_0 = 2$ and $t_n = 1$ for $n > 0$. Let us also define $l_n = 0$ for $n < 0$.

Then we get

Lemma 2

$$\sum_{k \geq 0} (-1)^k \binom{n}{k} l_{n-2k} = [n \geq 0]. \tag{7}$$

Proof

We prove this by induction. It is true for $n = 0$ and $n = 1$. Assume it holds for $2n - 1$. Then we get

$$1 = \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{k} l_{2n-1-2k} = \sum_{k=0}^{n-1} (-1)^k \binom{2n-1}{k} \left(l_{2n-2k} - t_{2n-2-2k} l_{2n-2-2k} \right)$$
$$= l_{2n} + \sum_{k=1}^{n-1} (-1)^k \binom{2n-1}{k} l_{2n-2k} - \sum_{k=0}^{n-2} (-1)^k \binom{2n-1}{k} l_{2n-2k-2} - (-1)^{n-1} 2 \binom{2n-1}{n-1} l_0$$
$$= l_{2n} + \sum_{k=1}^{n-1} (-1)^k \left(\binom{2n-1}{k} + \binom{2n-1}{k-1} \right) l_{2n-2k} + (-1)^n \frac{2n}{n} \binom{2n-1}{n-1} l_0$$
$$= \sum_{k=0}^{n} (-1)^k \binom{2n}{k} l_{2n-2k}.$$

Finally suppose it is true for $2n$. Then we get

$$1 = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} l_{2n-2k} = \sum_{k=0}^{n} (-1)^k \binom{2n}{k} \left(l_{2n+1-2k} - l_{2n-1-2k} \right)$$
$$= l_{2n+1} + \sum_{k=1}^{n} (-1)^k \binom{2n}{k} l_{2n+1-2k} - \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} l_{2n-2k-1}$$
$$= l_{2n+1} + \sum_{k=1}^{n} (-1)^k \left(\binom{2n}{k} + \binom{2n}{k-1} \right) l_{2n+1-2k}$$
$$= \sum_{k=0}^{n+1} (-1)^k \binom{2n+1}{k} l_{2n+1-2k}.$$

After these preliminaries we can compute the above determinants.

Combining the identities $\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}l_{2n+1-2k}=1$ and

$$\sum_{k=0}^{n}(-1)^k\binom{2n-1}{k}l_{2n-1-2k}=\sum_{k=1}^{n}(-1)^{k+1}\binom{2n-1}{k-1}l_{2n+1-2k}=1$$

we get

$$\sum_{k=0}^{n}(-1)^k\left(\binom{2n+1}{k}+\binom{2n-1}{k-1}\right)l_{2n+1-2k}=[n=0]$$ or equivalently

$$\sum_{k=0}^{n}(-1)^{n-k}\left(\binom{2n+1}{n-k}+\binom{2n-1}{n-k-1}\right)l_{2k+1}=[n=0]$$

By (5) and (6) this implies (1).

For the second determinant we get from $\sum_{k=0}^{n}(-1)^k\binom{2n}{k}l_{2n-2k}=1$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n-k}L_{2k}=1+(-1)^n\binom{2n}{n}. \tag{8}$$

Let $r(n):=\sum_{j=0}^{n}(-1)^{n+j}\binom{2n}{n+j}=\sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{n-j}=\sum_{j=0}^{n}(-1)^j\binom{2n}{j}$.

Then

$$2r(n)=(-1)^n\binom{2n}{n}+\sum_{j=0}^{2n}(-1)^j\binom{2n}{j}=(1-1)^{2n}+(-1)^n\binom{2n}{n}=[n=0]+(-1)^n\binom{2n}{n},$$

Together with (8) we get

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{n-k}\left(L_{2k}-2\right)=1-[n=0]. \tag{9}$$

This identity with $n+1$ in place of n gives

$$\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{2n+2}{n+1-k}\left(L_{2k}-2\right)=1. \tag{10}$$

Subtracting (9) from (10) gives

$$\sum_{k=0}^{n+1}(-1)^{n+1-k}\left(\binom{2n+2}{n+1-k}+\binom{2n}{n-k}\right)\left(L_{2k}-2\right)=[n=0]$$

or equivalently since $L_0-2=0$

$$\sum_{k=0}^{n}(-1)^{n-k}\left(\binom{2n+2}{n-k}+\binom{2n}{n-k-1}\right)\left(L_{2k+2}-2\right)=[n=0] \qquad (11)$$

This implies

$$\det\left(\binom{2i+2}{i+j+2}+\binom{2i+4}{i+j+3}\right)_{i,j=0}^{n-1}=\det\left(\binom{2i+2}{i-j}+\binom{2i+4}{i-j+1}\right)_{i,j=0}^{n-1}=L_{2n+2}-2. \quad (12)$$

To obtain (2) we must verify that $\sum_{j=0}^{n}L_{2j+1}=L_{2n+2}-2.$

This is easily shown by induction: $2+L_1=L_0+L_1=L_2$.

If we know already that $2+\sum_{j=0}^{n-1}L_{2j+1}=L_{2n}$, then we get

$$2+\sum_{j=0}^{nn}L_{2j+1}=L_{2n}+L_{2n+1}=L_{2n+2}.$$

Consider now the Fibonacci numbers F_n with initial values $F_0=0$ and $F_1=1$. Let $f_n=F_n$ for $n\geq 0$ and $f_n=0$ for $n<0$.

Then we have $l_n=f_{n+1}+f_{n-1}$ for all integers n.

From $\sum_{k\geq 0}(-1)^k\binom{n}{k}l_{n-2k}=[n\geq 0]$ we get $\sum_{k\geq 0}(-1)^k\left(\binom{n}{k}-\binom{n}{k-1}\right)f_{n+1-2k}=[n\geq 0]$ since

$$[n\geq 0]=\sum_{k\geq 0}(-1)^k\binom{n}{k}\left(f_{n+1-2k}+f_{n-1-2k}\right)=\sum_{k\geq 0}(-1)^k\binom{n}{k}f_{n+1-2k}-\sum_{k\geq 0}(-1)^k\binom{n}{k-1}f_{n+1-2k}$$
$$=\sum_{k\geq 0}(-1)^k\left(\binom{n}{k}-\binom{n}{k-1}\right)f_{n+1-2k}.$$

From $\sum_{k=0}^{n}(-1)^k\left(\binom{2n}{k}-\binom{2n}{k-1}\right)f_{2n+1-2k}=\sum_{k=0}^{n}(-1)^{n-k}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)f_{2k+1}=\left[n\geq 0\right]$

and $\sum_{k=1}^{n}(-1)^{n-k-1}\left(\binom{2n-2}{n-1-k}-\binom{2n-2}{n-k-2}\right)f_{2k+1}=[n\geq 1]$ we get

$$f_1+\sum_{k=1}^{n}(-1)^{n-k}\left(\binom{2n-2}{n-1-k}-\binom{2n-2}{n-k-2}+\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)f_{2k+1}=[n=0]$$

and therefore

$$\det\left(\binom{2i}{i+j}-\binom{2i}{i+j+1}+\binom{2i+2}{i+j+1}-\binom{2i+2}{i+j+2}\right)_{i,j=0}^{n-1}=F_{2n+3}.$$

From

$$\sum_{k=0}^{n}(-1)^{k}\left(\binom{2n+1}{k}-\binom{2n+1}{k-1}\right)f_{2n+2-2k}=\sum_{k=0}^{n}(-1)^{n-k}\left(\binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}\right)f_{2k+2}=[n\geq 0]$$

and $\sum_{k=1}^{n}(-1)^{n-k-1}\left(\binom{2n-1}{n-1-k}-\binom{2n-1}{n-k-2}\right)f_{2k+2}=[n\geq 1]$ we get

$$f_2+\sum_{k=1}^{n}(-1)^{n-k}\left(\binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}+\binom{2n-1}{n-1-k}-\binom{2n-1}{n-k-2}\right)f_{2k+2}=[n=0]$$

and therefore we get

$$\det\left(\binom{2i+3}{i+1-j}-\binom{2i+3}{i-j}+\binom{2i+1}{i-j}-\binom{2i+1}{i-j-1}\right)_{i.j=0}^{n-1}=F_{2n+2}.$$